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ON THE

INSCRIPTION

OF A

REGULAR POLYGON OF *SEVENTEEN* SIDES IN A CIRCLE:

OR DIVISION OF THE CIRCUMFERENCE INTO SEVENTEEN EQUAL PARTS.

BY SAMUEL JAMES, ESQ.

PRESENTED BY THE REV. FRANCIS SADLEIR, D.D. F.T.C.D. M.R.I.A.

Read, January 24, 1820.

THE inscription of all regular polygons in a circle, besides those of which the constructions are given in the elements of Euclid, and those arising from them by continual bisections, or taking their differences, has been thought for ages to exceed the powers of elementary geometry; and, through the progress of mathematical discovery from the time of the ancient Greek geometricians, no addition to the number, as known to them, appears to have been made until about the beginning of the present century; when an unexpected discovery was made by *M. Gauss* of certain other regular polygons, which yet admitted of being inscribed geometrically in a circle. This discovery was announced to the public in a work published by him at Leipsic in the year 1801, entitled, *Disquisitiones*

Arithmeticæ, in which he has shown that all such regular polygons as have the number of their sides expressed by $2^n + 1$, and being at the same time a prime number, admit of being inscribed geometrically in a circle: and hence since 5, 17, 257, 65537, are prime numbers of this form, consequently regular polygons of these sides, admit of geometrical construction.

As the construction, however, of none of these polygons, except of the first, or regular pentagon, has yet been publicly given, I beg leave to submit that for the second, or a regular polygon of 17 sides, together with the analysis. The former will be found exceedingly simple, but as the latter depends on the application of some general properties of the circle, I shall previously demonstrate these in the four following *Lemmas*.

SAMUEL JAMES.

Dublin, 17*th October*, 1819.

LEMMA I.

If on each side of a given point in the circumference of a circle, equal arches be taken, parallel right lines drawn from their extremities, to terminate in a diameter passing through the given point, are equal to each other. *See Plate, Fig.* 1.

Let AB and AC be equal arches taken at each side of the given point A in the circumference BAC; and BD and CE two right lines drawn from their extremities B and C parallel to each other, terminating in the diameter FA passing through A: then will BD and CE be equal to each other.

For if BC be drawn cutting the diameter FA in I, it is well known, because the diameter FA bisects the arch BC, in A, that it also bisects its chord BC, in I; and because BC meets the parallels BD, CE in B and C, the angle IBD is equal to ICE, (29 B. I. Euc.) and for a like reason, because FA meets the parallels, the angle BDI is equal to CEI; therefore in the two triangles BDI and CEI, the two angles IBD, BDI, are equal to the two ICE, CEI, and the side BI is equal to IC, therefore the two triangles are equal in all respects, (26, B. I. Euc.) and the side BD to CE*.—Q. E. D.

LEMMA II.

If the semi-circumference of a circle be divided into any odd number of equal arches, and chords be drawn connecting the opposite points of division, the difference between the sums of those chords, alternately taken, will be equal to the radius of the circle. *See Plate, Fig.* 2.

Let the semi-circumference AEK be divided into any odd number of equal arches AB, BC, &c. and chords BI, CH, &c. be drawn, connecting the opposite points of division B, I; C, H, &c.; then will the difference between the sums of those chords, alternately taken, be equal to the radius of the circle; that is

.. $(BI + DG) - (CH + EF) = OK$, the radius of the circle.

For draw the radius OF to either extremity, as F, of the least chord EF, cutting the several other chords in the points L. M. N.

Then because the diameter AK, and chords BI, CH, &c. intercept the equal arches AB, KI; BC, IH; &c. they are parallel to each other, as is well known, and therefore, because the arch EF = FG, the right line EF is = NG, by the foregoing Lemma, and for the same reason DN is = MH, and CM = LI, &c.

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* The demonstration of this property might conveniently be made a part of that in the 2d Lemma; I have however, for the sake of distinctness, given it a separate demonstration.

$$\begin{array}{rrcl} \text{Hence therefore since} & NG & = & EF \\ & DN & = & MH \\ & LI & = & CM \\ \text{and} \ldots & BL & = & OK \\ \hline \end{array}$$

by addition . . $BI + DG = OK + CH + EF$

or . . $(BI + DG) - (CH + EF) = OK$ . . Q. E. D.

And in the same manner whatever be the number of equal arches.

Corollary.—Hence as the above chords are the chords subtending respectively 1, 3, 5, &c. of the equal arches, if therefore the semi-circumference be divided into any odd number of equal arches, the difference between the sums of the chords alternately taken, originating in the same point, and subtending 1, 3, 5, &c. of those arches, is equal to the radius of the circle: the greater sum being always that in which the greatest chord is included.

LEMMA III.

In a circle, the rectangle under the radius and the sum of the chords of two arches, is equal to the rectangle under the chord of half their sum, and the supplemental chord of half their difference: and,

LEMMA IV.

The rectangle under the radius and the difference of the chords of two arches, is equal to the rectangle under the supplemental chord of half their sum, and chord of half their difference. See *Plate, Fig. 3.*

Let AB and ABD be two arches of which AB, and AD are their chords; bisect the arch BD in C, and join AC, BC, CD; take O the centre of the circle, and draw the diameter COE, and join AE, BE, DE.

Then because ACDE is a quadrilateral inscribed in the circle, by Ptolemy's Theorem, (Prop. D. Book VI. Simson's Euclid.)

$$AD \times CE = AC \times DE + CD \times AE.$$

$$\left.\begin{array}{l}\text{or because } DE = BE\\ \text{and . . } \quad CD = BC\end{array}\right\} \therefore AD \times CE = AC \times BE + BC \times AE \ldots \text{I.}$$

Again, because ABCE is a quatrilateral inscribed in the circle.

$$AC \times BE = AB \times CE + BC \times AE$$

$$\text{or} \quad AB \times CE = AC \times BE - BC \times AE \; .. \text{II.}$$

$$\left.\begin{array}{l}\text{and hence by addition}\\ \quad \text{I. and II. . . .}\end{array}\right\} \therefore (AD + AB) \times CE = 2\,AC \times BE;$$

$$\text{or because } CE = 2\,CO \,.\, (AD + AB) \times 2\,CO = 2\,AC \times BE;$$

or . $(AD + AB) \times CO = AC \times BE$; which is the 3d Lemma: for CO is the radius of the circle, AD + AB the sum of the chords of the two arches ABD and AB; AC the chord of half their sum ABC, and BE the supplemental chord of half their difference BC, or CD; whence the truth of the 3d Lemma is manifest.

In the same manner, by subtraction, I. and II.

$(AD - AB) \times CO = BC \times AE$; which is the 4th Lemma: for CO is the radius of the circle, AD — AB the difference of the chords of the two arches ADB and AB; AE the supplemental chord of half their sum, and BC the chord of half their difference: whence the truth of the 4th Lemma is also manifest.

ANALYSIS.—*See Plate, Fig.* 4.—Suppose now in the semicircle AEK, the semicircumference to be divided into seventeen equal arches, of which the points B, C, D, &c. are the 1st, 3d, 5th, &c. points of division, and AB, AC, AD, &c. the chords drawn from A, the extremity of the diameter, to those points respectively.

Then agreeably to the Corollary to the II. Lemma, the difference between the sums of those chords, alternately taken, is equal to the radius of the circle; that is . .

$$(AI+AG+AE+AC)-(AH+AF+AD+AB)=AO$$

or . $(AG+AE+AC-AD)-(AH+AB+AF-AI)=AO$. Theor. I.

But a remarkable property of these chords, $(AG+AE+AC-AD)$ and $(AH+AB+AF-AI)$ will be found to result from the application of the 3d and 4th Lemmas, viz.

The sum of all the rectangles under each of the first four, and each of the four following, is equal to four times the square of the radius of the circle.—(See Note 1.)

That is, . $(AG+AE+AC-AD)\times(AH+AB+AF-AI)=4\,AO^2$ . . . Theor. II.

Again, by the application of the same Lemmas, it will appear that . . $(AH+AB)\times(AI-AF)=AO^2$ . . Theor. III. *See Note 2.*
and . . $(AG+AE)\times(AD-AC)=AO^2$ . . Theor. IV. *See Note 3.*

$$\left.\begin{array}{r}\text{And by Lemma}\\ \text{4th . . . that}\end{array}\right\}\ AO\times(AD-AC)=AH\times AB\ \text{. . . Theor. V.}$$

Since therefore in Theorems I. and II. are given

$$\left.\begin{array}{l}(AG+AE+AC-AD)-(AH+AB+AF-AI)=AO\\ \text{and }(AG+AE+AC-AD)\times(AH+AB+AF-AI)=4\,AO^2\end{array}\right\}\text{A.}$$

by the construction of a well known simple problem of finding two lines of which their *difference* and *rectangle* are given,

$$\left.\begin{array}{r}AG+AE+AC-AD\\ \text{and . . . . }AH+AB+AF-AI\end{array}\right\}\text{ are given.}$$

but . . . . $AH+AB+AF-AI$
is the same as . . $(AH+AB)-(AI-AF)$

$$\left.\begin{array}{l}\text{therefore . . . . . }(AH+AB)-(AI-AF)\text{ is given}\\ \text{and Theor. III. . }(AH+AB)\times(AI-AF)\text{ is given}\end{array}\right\}\text{ . . . . B.}$$

$$\left.\begin{array}{l}\text{therefore, as before }AH+AB\\ \text{and . . . . }AI-AF\end{array}\right\}\text{ are given.}$$

$$\left.\begin{array}{ll}\text{Again since in Analysis A .} & (AG + AE)-(AD-AC) \text{ is given}\\ \text{and Theor. IV. . . . . . .} & (AG + AE) \times (AD-AC) \text{ is given}\end{array}\right\}\text{ C.}$$

$$\begin{array}{ll}\text{therefore . . . .} & \left.\begin{array}{l}AG + AE\\ AD-AC\end{array}\right\}\text{ are given ;}\\ \text{and . . . .} & \end{array}$$

and thus since . . . . $AD - AC$ is given, and AO is given,

$(AD - AC) \times AO$ is given;

that is Theor. V. . . . . $AH \times AB$ . . is given,

but Analysis B. . . . . $AH + AB$ . . is given;

therefore the *sum* and *rectangle* of AH and AB are given, whence AH and AB are each given, and thence, as is evident, the side of the polygon.

And hence now, combining the several constructions as derived in the Analysis, the construction of the problem, or division of the circumference into 17 equal parts, will be simply as follows:—

CONSTRUCTION.—*See Plate, Fig.* 5.—Draw the radius OC perpendicular to the diameter AK; take OD equal to one-fourth of the radius; and make DE, DF, each equal to DC; and EG and FH equal to EC and FC respectively; divide OG into two parts OI, IG, such, that their rectangle shall be equal to $OH \times AO$; then if from A, OI be applied on the circumference to R, and the arch KR be bisected in P — or IG be taken twice in the circumference from A to N; the arch KP, or AN will be the 17th part of the whole circumference.

For if DC, EC, FC be drawn, it is well known, from the simple construction of finding two lines of which their *difference* and *rectangle* are given,

that $DC + DO$	$EC + EO$	$FC + FO$	will be the greater
and $DC - DO$	$EC - EO$	$FC - FO$	. . . . . the less

of two lines of which *half their difference* is DO, EO and FO respectively, and *rectangle* OC^2.

And hence it is evident that in Analysis A, where half the difference of the chords, as specified, is $= 2OD$, and their rectangle $4AO^2$, or $4OC^2$,

$$\left.\begin{array}{l} \text{that . } AG+AE+AC-AD \text{ is } = 2(DC+DO)=2OF \\ \text{and . } AH+AB+AF-AI\ .\ .\ \ \ 2(DC-DO)=2OE \end{array}\right\}\text{ by construction;}$$

and therefore now Analysis B

$$(AH+AB)-(AI-AF) \text{ is given } = 2OE$$

and . . . . $(AH+AB)\times(AI-AF) = AO^2 = OC^2$.

and Analysis C $(AG+AE)-(AD-AC) = 2OF$

and . . . . $(AG+AE)\times(AD-AC) = AO^2 = OC^2$.

Whence under the construction referred to

$$\left.\begin{array}{l} \text{there is given in the former } AH+AB=EC+EO=OG, \\ \text{and in the latter . . . . } AD-AC=FC-FO=OH \end{array}\right\}\text{ by construction.}$$

and hence Theor. V. . . $AH\times AB=(AD-AC)\times AO=OH\times AO$

and therefore finally . . . . $AH+AB=OG$

and . . . . $AH\times AB=OH\times AO$

But since OG has been divided into two parts OI, IG, the rectangle of which is equal to $OH\times AO$, consequently OI is $=$ AH, and IG $=$ AB, but AH is *the supplemental chord of twice the arch*, and AB *the chord of half the arch*, subtended by the side of the polygon, by hypothesis; and hence, OI and IG are equal to those chords, respectively, and since OI has been applied on the circumference from A to R, and the arch KR bisected in P; and IG taken twice on the circumference from A to N, it is evident that the arches KP and AN are each the 17th part of the whole circumference.

COROLLARY.—IG is the side of a regular polygon of 34 sides inscribed in the circle.

SCHOLIUM.—Such is the construction of this interesting problem to which we are led by the preceding Analysis: a construction in itself exceedingly simple, falling little short of the well known simple construction of a regular Pentagon in a Circle; the principles, however, from which this construction is derived, are less evident, involving some peculiar properties of the chords subtending 1, 3, 5, &c. of the equal arches arising from the division of the semicircumference into *seventeen* equal parts, or which is the same, of the Supplemental Chords of the progressive arches arising from the division of the whole circumference into 17 equal parts. These properties form the five Theorems in the Analysis, and which comprehend all the necessary data for the solution of the problem. The 1st is an obvious deduction from the 2d Lemma, and the 5th, the immediate application of the 4th:—the 2d, 3d and 4th properties are derived by a more extended application of the 3d and 4th Lemmas, as shewn in the notes; of these, the 2d property is very remarkable, and of great importance in enabling us at the commencement of the Analysis, combined with Theorem I. to effect the decomposition of the chords as therein specified; and without which we might perhaps in vain attempt the solution:—the 3d and 4th properties are equally valuable in effecting a further decomposition of these chords; leading finally to the entire separation of the supplemental chord of twice the arch, and chord of half the arch subtended by the side of the Polygon, and thence, by an obvious construction, to the side of the Polygon: and it might perhaps be worth while to enquire how far the principles employed in this Analysis might be attended with advantage in their application to the solution of similar problems.

The five Theorems in the Analysis may be conveniently exhibited under the following abridged form:

$$\left.\begin{array}{rl} \text{Put}\quad AG+AE &= x\\ AD-AC &= y\\ AH+AB &= z\\ AI-AF &= v\\ \text{and}\ \ldots\ AO &= r \end{array}\right\}\ \text{Then}\ \left.\begin{array}{l} (x-y)-(z-v)= r\\ (z-y)\times(z-v)=4r^2\\ zv\ \ldots\ldots\ldots\ldots = r^2\\ xy\ \ldots\ldots\ldots\ldots = r. \end{array}\right\}\ \begin{array}{l}\text{whence } z \text{ and } y\\ \text{are determined;}\end{array}$$

$$\left.\begin{array}{rl} \text{then}\ AH+AB &= z\\ \text{and}\ AH\times AB &= y \end{array}\right\}\ \text{Whence AH and AB are determined.}$$

I shall conclude this paper with the following numerical calculation of the length of the side of the Polygon:

Let the Radius AO = 1; and suppose the chords AR, AP, KP, and AN, to be drawn; then following the several constructions,

$$OD=\tfrac{1}{4},\ DC=\sqrt{OC^2+OD^2}=\sqrt{1+\tfrac{1}{16}}=\tfrac{1}{4}\sqrt{17};$$

$$OF=DC+DO=\tfrac{1}{4}\sqrt{17}+\tfrac{1}{4}=\tfrac{1}{4}(\sqrt{17}+1;)$$

$$FC=\sqrt{OF^2+OC^2}=\tfrac{1}{4}\sqrt{34+2\sqrt{17}};$$

$$OH=FC-OF=\tfrac{1}{4}(\sqrt{34+2\sqrt{17}}-\sqrt{17}-1);$$

$$\text{Again } EO=DC-DO=\tfrac{1}{4}(\sqrt{17}-1);$$

$$EC=\sqrt{EO^2+OC^2}=\tfrac{1}{4}\sqrt{34-2\sqrt{17}},$$

$$\text{and } OG=EC+EO=\tfrac{1}{4}(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1)$$

$$\text{and hence } OI+IG\ .\ (OG)=\tfrac{1}{4}(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1)$$

$$\text{and } OI\times IG+(OH\times AO)=\tfrac{1}{4}(\sqrt{34+2\sqrt{17}}-\sqrt{17}-1)\times 1.$$

Whence, by the well known division of a right line into two parts containing a given rectangle, we shall have

$$OI=\tfrac{1}{2}OG+\sqrt{\tfrac{1}{2}OG^2-OH\times AO}$$

$$\text{and } IG=\tfrac{1}{2}OG-\sqrt{\tfrac{1}{2}OG^2-OH\times AO}$$

which in numbers gives OI, the supplemental chord of *twice* the arch subtended by the side of the Polygon, =

$$\tfrac{1}{8}\left(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1\right)$$

$$+\sqrt{\left(\tfrac{1}{8}\left(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1\right)\right)^2-\tfrac{1}{4}\left(\sqrt{34+2\sqrt{17}}-\sqrt{17}-1\right)}=$$

(by reduction $\tfrac{1}{8}\left(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1\right)$

$$+\tfrac{1}{4}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}=AR:$$

but by a well known Theorem, " If the supplemental chord of an " arch be encreased by the number 2, (the radius being unity) the " square root of the sum will be the supplemental chord of half " that arch," and hence

$$\overline{AP}^2=2+\tfrac{1}{8}\left(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1\right)$$

$$+\tfrac{1}{4}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}},$$

and therefore

$$\overline{KP}^2\left(=\overline{AK}^2-\overline{AP}^2\right)=2-\tfrac{1}{8}\left(\sqrt{34-2\sqrt{17}}+\sqrt{17}-1\right)$$

$$-\tfrac{1}{4}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}=$$

$$\tfrac{1}{8}\Big(17-\sqrt{17}-\sqrt{34-2\sqrt{17}}-$$

$$2\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}\Big),$$

or

$$KP=\sqrt{\tfrac{1}{8}\Big(17-\sqrt{17}-\sqrt{34-2\sqrt{17}}}$$

$$\overline{-2\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}\Big)},$$

which, reduced to numbers, gives KP or AN, the side of the polygon=.367499035633, &c.

NOTE I.

This property is thus derived.

From the third and fourth Lemmas it will appear that...

$$\begin{aligned}
AG \times AH &= AO \times \overline{AI+AE} \\
AG \times AB &= AO \times \overline{AE-AD} \\
AG \times AF &= AO \times \overline{AI+AC} \\
AG \times -AI &= AO \times -\overline{AH-AF}
\end{aligned}$$

$$\begin{aligned}
AE \times AH &= AO \times AG+AC \\
AE \times AB &= AO \times AG-AF \\
AE \times AF &= AO \times AI-AB \\
AE \times -AI &= AO \times -AF-AD
\end{aligned}$$

$$\begin{aligned}
AC \times AH &= AO \times AE-AB \\
AC \times AB &= AO \times AI-AH \\
AC \times AF &= AO \times AG-AD \\
AC \times -AI &= AO \times -AD-AB
\end{aligned}$$

$$\begin{aligned}
-AD \times AH &= AO \times -AF-AB \\
-AD \times AB &= AO \times -AH+AG \\
-AD \times AF &= AO \times -AH+AC \\
-AD \times -AI &= AO \times AE+AC.
\end{aligned}$$

Now in the above rectangles under the radius and the sum, or difference of the chords as specified, it will be observed that each chord is *four times* taken; whence by addition it is evident, that these rectangles will become . . .

$AO \times (4AG+4AE+4AC-4AD)-(4AH+4AB+4AF-4AI)$; that is by Lemma II. $AO \times 4AO$ or $4AO^2$, whence the truth of this property is manifest.

NOTE II.

This property is derived in a similar manner.

$$\left.\begin{array}{l} AH \times AI = AO \times AG + AI \\ AH \times -AF = AO \times -AD - AH \\ \\ AB \times AI = AO \times AC - AB \\ AB \times -AF = AO \times AE - AF \end{array}\right\} \begin{array}{l} \text{whence by addition} \\ (AH+AB) \times (AI-AF) \\ = AO \times (AG+AE+AC-AD)- \\ (AH+AB+AF-AI) \\ = \text{(by Lemma II.) } AO \times AO, \text{ or } AO^2. \end{array}$$

NOTE III.

In like manner this property is derived.

$$\left.\begin{array}{l} AG \times AD = AO \times AG - AB \\ AG \times -AC = AO \times AC - AF \\ \\ AE \times AD = AO \times AI - AD \\ AE \times -AC = AO \times AE - AH \end{array}\right\} \begin{array}{l} \text{whence by addition} \\ (AG+AE) \times (AD-AC) \\ = AO \times (AG+AE+AC-AD)- \\ (AH+AB+AF-AI) \\ = \text{(by Lemma II.) } AO \times AO, \text{ or } AO^2. \end{array}$$